Title: On sums of strictly narrow operators acting from a Riesz Space to a Banach Space

Author: Oleksandr Maslyuchenko, Mikhail Popov

Hindawi
Journal of Function Spaces
Volume 2019, Article ID 8569409, 6 pages
https://doi.org/10.1155/2019/8569409

Research Article

On Sums of Strictly Narrow Operators Acting from a Riesz Space to a Banach Space

Oleksandr Maslyuchenko[1,2] and Mikhail Popov[3,4]

[1] *Institute of Mathematics, University of Silesia in Katowice, Bankowa 12, 40-007 Katowice, Poland*
[2] *Yuriy Fedkovych Chernivtsi National University, Department of Mathematical Analysis, Kotsiubynskoho 2, 58012 Chernivtsi, Ukraine*
[3] *Institute of Mathematics, Pomeranian University in Słupsk, ul. Arciszewskiego 22d, 76-200 Słupsk, Poland*
[4] *Vasyl Stefanyk Precarpathian National University, Ukraine*

Correspondence should be addressed to Mikhail Popov; misham.popov@gmail.com

Received 6 January 2019; Revised 30 April 2019; Accepted 19 May 2019; Published 2 June 2019

Academic Editor: Miguel Martin

We prove that if E is a Dedekind complete atomless Riesz space and X is a Banach space, then the sum of two laterally continuous orthogonally additive operators from E to X, one of which is strictly narrow and the other one is hereditarily strictly narrow with finite variation (in particular, has finite rank), is strictly narrow. Similar results were previously obtained for narrow operators by different authors; however, no theorem of the kind was known for strictly narrow operators.

1. Introduction

Narrow operators were introduced in 1990 [1]; however, some deep results on these operators were obtained earlier; see [2]. Generalizing compact operators on function spaces, narrow operators gave new geometric facts. The most unusual thing about narrow operators is that, on the space L_1, the sum of two continuous linear narrow operators is narrow [2, Theorem 7.46]; however, if a rearrangement invariant space E on $[0, 1]$ has an unconditional basis, then every operator on E is a sum of two narrow operators [2, Theorem 5.2].

A result of Mykhaylyuk and the second named author asserts that, for every Köthe Banach space E on $[0, 1]$, there exist a Banach space X and narrow operators from E to X with nonnarrow sum [3].

If the norm of the domain Köthe Banach space E is not absolutely continuous (for instance, if $E = L_\infty$), then the usual technique does not work. So, there are nonnarrow continuous linear functionals on L_∞. However, questions about narrowness of the sum of two narrow operators are still interesting. A sum of two narrow operators on L_∞ need not be narrow [4]. Moreover, if $1 < p \le \infty$, then there are regular narrow operators $S, T : L_p \longrightarrow L_\infty$ with nonnarrow sum $S+T$ [3].

Now let a pair of spaces E, X be such that there are narrow operators $S, T : E \longrightarrow X$ with nonnarrow sum $S + T$. Is the sum of a narrow operator and a compact (or even finite rank) operator narrow? It is known [2, Corollary 11.4] that if E is a Köthe Banach space with an absolutely continuous norm, then for any Banach space X the sum of a narrow operator and a "small" operator (like compact, AM-compact, Dunford-Pettis operators, etc.) is narrow.

If the norm of E is not absolutely continuous and a compact operator need not be narrow, a weaker question naturally arises: is the sum of two narrow operators, at least one of which is compact, narrow? The strongest result in this direction was obtained by Mykhaylyuk [5]: if E is a Köthe F-space, X is a locally convex F-space, and $S, T \in \mathscr{L}(E, X)$ are narrow operators such that T maps the set of all signs to a relatively compact subset of X (in particular, if T is compact), then the sum $S + T$ is narrow.

In 2014, narrow operators were generalized to nonlinear maps, more precisely to orthogonally additive operators [6], which were studied by Mazón, S. Segura de León in [7, 8]. In different contexts, when dealing with narrow linear operators, the linearity has been used for orthogonal pairs of elements only. This allowed generalizing results on narrow operators obtained in [9] from linear to orthogonally additive operators. For example, a result of [6] asserts that every laterally continuous C-compact orthogonally additive operator acting from an atomless Dedekind complete Riesz space is narrow. Recently, the latter theorem was essentially generalized in [10] by proving that if E is a Dedekind complete atomless Riesz space and X is a Banach space, then the sum of narrow and C-compact laterally continuous orthogonally additive operators from E to X is narrow.

However, no result is known concerning a sum of two strictly narrow operators. Notice that in every known example of two narrow operators with nonnarrow sum, the summands are not strictly narrow. To be more precise, we recall necessary definitions.

By a *Köthe Banach space* on a finite atomless measure space (Ω, Σ, μ), we mean a Banach space E which is a linear subspace of $L_1(\mu)$ possessing the following properties: $\mathbf{1}_\Omega \in E$, and for every $x \in E$ and $y \in L_1(\mu)$ the condition $|y| \le |x|$ implies that $y \in E$ and $\|y\| \le \|x\|$ (by $\mathbf{1}_A$ we denote the characteristic function of a set $A \in \Sigma$, and the inequality $u \le v$ in E means that $\tilde{u}(t) \le \tilde{v}(t)$ holds for μ-almost all $t \in \Omega$, where $\tilde{u} \in u$ and $\tilde{v} \in v$ are some/any representatives of the classes u, v). A Köthe Banach space E on a finite atomless measure space (Ω, Σ, μ) is said to have an *absolutely continuous norm* if $\lim_{\mu(A) \longrightarrow 0} \|x \cdot \mathbf{1}_A\| = 0$ for all $x \in E$.

If X, Y are Banach spaces, by $\mathscr{L}(X, Y)$ we denote the Banach space of all continuous linear operators $T : X \longrightarrow Y$, and $\mathscr{L}(X)$ stands for $\mathscr{L}(X, X)$. By $x \sqcup y$ we denote the disjoint sum $x + y$ in a Köthe Banach space, that is, under the assumption $\operatorname{supp} x \cap \operatorname{supp} y = \emptyset$ or, more generally, in a Riesz space under the assumption $x \perp y$. In a Boolean algebra, $x \sqcup y$ means the disjoint supremum $x \vee y$, that is, under the assumption $x \wedge y = \mathbf{0}$.

For familiarly used information on Riesz spaces, the reader can refer to [11]. Let E be a Riesz space and X be a linear space. A function $T : E \longrightarrow X$ is called an *orthogonally additive operator* (*OAO* in short) if $T(x \sqcup y) = T(x) + T(y)$ for all disjoint elements $x, y \in E$. Simple examples of OAOs are the positive, negative parts and the modules of an element: $T_1(x) = x^+$, $T_2(x) = x^-$, $T_3(x) = |x|$, and $x \in E$. For more examples of OAOs including integral Uryson operators, see [6–8].

An element x of a Riesz space E is called a *fragment* of $y \in E$ (write $x \sqsubseteq y$) provided $x \perp y - x$. The set of all fragments of an element $e \in E$ is denoted by $\mathfrak{F}_e$. Observe that if $z = x \sqcup y$, then x and y are disjoint fragments of z. We say that an element $a \neq 0$ of a Riesz space E is an *atom* if the only fragments of a are 0 and a itself. A Riesz space having no atom is said to be *atomless*.

Let E be an atomless Riesz space and let X be a Banach space. An OAO $T : E \longrightarrow X$ is called

(i) *narrow at a point* $e \in E$ if for every $\varepsilon > 0$ there is a decomposition $e = e' \sqcup e''$ such that $\|T(e') - T(e'')\| < \varepsilon$;

(ii) *narrow* if it is narrow at each point $e \in E$;

(iii) *strictly narrow at a point* $e \in E$ if there is a decomposition $e = e' \sqcup e''$ such that $T(e') = T(e'')$;

(iv) *strictly narrow* if it is strictly narrow at each point $e \in E$.

The atomlessness assumption in the above definition serves to avoid triviality, because otherwise every narrow or strictly narrow operator must send an atom to zero.

Observe that $T(0) = 0$ for every OAO T; hence, every OAO is strictly narrow at zero. Every strictly narrow (at a point e) is narrow (at a point e); however, the converse is not true [2, Proposition 2.2]. Under mild assumptions on the domain Riesz space, every operator with finite-dimensional range is strictly narrow and every operator from an atomless Banach lattice to a purely atomic Banach lattice is strictly narrow [12].

If E is a Köthe Banach space with an absolutely continuous norm on a finite atomless measure space (Ω, Σ, μ) and X is a Banach space, then an OAO $T : E \longrightarrow X$ is narrow if and only if for every $\varepsilon > 0$ every $A \in \Sigma$ admits a decomposition $A = B \sqcup C$, $B, C \in \Sigma$ such that $\mu(B) = \mu(C)$ and $\|T(\mathbf{1}_B) - T(\mathbf{1}_C)\| < \varepsilon$ [2, Proposition 10.2], and a similar statement holds for strictly narrow OAOs. Remark that the latter property of narrow (strictly narrow) operators was initially considered as a definition.

One more definition for Köthe Banach spaces is essential for our investigation. Let E be a Köthe Banach space on a finite atomless measure space (Ω, Σ, μ), and let X be a Banach space. An operator $T \in \mathscr{L}(E, X)$ is called *hereditarily narrow* if for every $A \in \Sigma$, $\mu(A) > 0$ and every atomless sub-σ-algebra $\mathscr{F}$ of $\Sigma(A)$ the restriction of T to $E(\mathscr{F})$ is narrow (here $E(\mathscr{F}) = \{x \in E(A) : x \text{ is } \mathscr{F} - \text{measurable}\}$). The following proposition gives lots of examples of pairs of narrow operators with narrow sum.

Proposition 1 ([13], [2], Proposition 11.2). *Let E be a Köthe Banach space on $[0,1]$ with an absolutely continuous norm, and let X be a Banach space. Then the sum $T = T_1 + T_2$ of a narrow operator $T_1 \in \mathscr{L}(E, X)$ and a hereditarily narrow operator $T_2 \in \mathscr{L}(E, X)$ is narrow. In particular, the sum of two hereditarily narrow operators is hereditarily narrow.*

Questions on the strict narrowness of sums of strictly narrow operators seem to be much more involved than similar questions on narrow operators. So, no example is known of strictly narrow operators with nonstrictly narrow sum.

Problem 2. Let E be an atomless Riesz space, and let X be a Banach space. Is the sum $S + T$ of strictly narrow operators $S, T : E \longrightarrow X$ strictly narrow or, at least, narrow?

Our main result, which is an analogue of Proposition 1 for strictly narrow operators, is the first result in this direction. The idea of the proof, inspired by paper [12], is to consider the set $\mathfrak{F}_e$ of all fragments of a fixed element of the domain Riesz space E as the main object for investigation. This becomes possible because the definitions of all notions from the main theorem could be equivalently restricted to $\mathfrak{F}_e$. Since the set $\mathfrak{F}_e$ is a Boolean algebra with respect to the natural operations, we come to analogous questions for functions defined on a Boolean algebra.

2. Dividing Measures on Boolean Algebras

Let (u_α) be a net in a Boolean algebra $\mathscr{B}$. The notation $u_\alpha \downarrow \mathbf{0}$ means that the net (u_α) decreases and $\inf_\alpha u_\alpha = \mathbf{0}$. We say that a net (x_α) in $\mathscr{B}$ *order converges* to an element $x \in \mathscr{B}$ if there exists a net (u_α) in $\mathscr{B}$ with the same index set such that $x_\alpha \triangle x \le u_\alpha$ for all indices α and $u_\alpha \downarrow \mathbf{0}$. In this case, we write $x_\alpha \longrightarrow x$ and say that x is the *order limit* of (x_α).

A Boolean algebra $\mathscr{B}$ is said to be *order complete* if any nonempty subset of $\mathscr{B}$ has the supremum. A Boolean algebra $\mathscr{B}$ is said to be *σ-complete* if any countable subset of $\mathscr{B}$ has the supremum. By a *partition* (of unity) in a Boolean algebra $\mathscr{B}$ we mean a maximal disjoint subset $\mathscr{A} \subseteq \mathscr{B}$, that is, $(\forall x \in \mathscr{B})$ $((\forall a \in \mathscr{A}\ a \cap x = \mathbf{0}) \Longrightarrow (x = \mathbf{0}))$. A *disjoint union* $\bigcup \mathscr{A}$ (i.e., the union of a disjoint system $\mathscr{A} \subseteq \mathscr{B}$), if exists, is denoted by $\coprod \mathscr{A}$. Although in some cases an infinite union in a Boolean algebra does not exist, it is immediate that if $\mathscr{A}$ is a partition then $\coprod \mathscr{A} = \mathbf{1}$ exists. Conversely, if $\coprod \mathscr{A} = \mathbf{1}$ then $\mathscr{A}$ is a partition.

Let $\mathscr{B}$ be a Boolean algebra, and let X be a linear space. A function $f : \mathscr{B} \longrightarrow X$ is said to be a *measure* provided $f(x \sqcup y) = f(x) + f(y)$ for every pair of disjoint elements $x, y \in \mathscr{B}$. Obviously, $f(\mathbf{0}) = 0$ for a measure. An element $a \in \mathscr{B}$ is called an *atom* of a measure $f : \mathscr{B} \longrightarrow X$ provided $f(a) \ne 0$ and for any $x \in \mathscr{B}$ with $x \le a$ one has either $f(x) = 0$ or $f(x) = f(a)$. A measure $f : \mathscr{B} \longrightarrow X$ is called *atomless* provided there is no atom of f.

A measure $f : \mathscr{B} \longrightarrow X$ is said to have *finite rank* if the closed linear span $[f(\mathscr{B})]$ is a finite-dimensional subspace of X. Let $\mathscr{B}$ be a σ-complete Boolean algebra, and let X be a Banach space. A measure $f : \mathscr{B} \longrightarrow X$ is said to be *σ-additive* provided for every disjoint sequence $(x_n)_{n=1}^\infty$ in $\mathscr{B}$ one has $f(\bigvee_{n=1}^\infty x_n) = \sum_{n=1}^\infty f(x_n)$, where the series converges unconditionally in X.

Let $\mathscr{B}$ be a Boolean algebra, and let X be a set. A function $f : \mathscr{B} \longrightarrow X$ is said to be *dividing* provided every element $b \in \mathscr{B}$ has a two-point partition $b = b' \sqcup b''$ with $f(b') = f(b'')$. We say that a pair of functions $f, g : \mathscr{B} \longrightarrow X$ is *uniformly dividing* if every element $b \in \mathscr{B}$ has a two-point partition $b = b' \sqcup b''$ with $f(b') = f(b'')$ and $g(b') = g(b'')$.

Next we define a hereditarily dividing measure, which takes an important place in our investigation. Given a Boolean algebra $\mathscr{B}$ and any $b \in \mathscr{B}$, we set $\mathscr{B}_b = \{x \in \mathscr{B} : x \le b\}$, which is a Boolean algebra with the induced operations and unity b.

Definition 3. Let $\mathscr{B}$ be a σ-complete Boolean algebra and let X be a Banach space. An atomless σ-additive measure $T : \mathscr{B} \longrightarrow X$ is called *hereditarily dividing* if, for every $b \in \mathscr{B}$ and every σ-complete subalgebra $\mathscr{U}$ of $\mathscr{B}_b$, the atomlessness of the restriction $T|_{\mathscr{U}}$ of T to $\mathscr{U}$ implies that $T|_{\mathscr{U}}$ is dividing on $\mathscr{U}$.

Obviously, a hereditarily dividing measure is dividing. By [12, Theorem 2.11] and Lemma 9, every atomless σ-additive measure with finite-dimensional range is dividing. Hence, as a consequence, we obtain the following example of hereditarily dividing measures.

Theorem 4. *Let $\mathscr{B}$ be a σ-complete Boolean algebra, and let X be a Banach space. Then every atomless σ-additive measure $T : \mathscr{B} \longrightarrow X$ with finite-dimensional range is hereditarily dividing.*

The following theorem brings an important tool for the main result.

Theorem 5. *Let $\mathscr{B}$ be a σ-complete Boolean algebra, and let X be a Banach space. Let $S, T : \mathscr{B} \longrightarrow X$ be σ-additive measures. If S is dividing and T is hereditarily dividing and has finite variation, then $S + T$ is dividing.*

Actually, we prove more.

Theorem 6. *Let $\mathscr{B}$ be a σ-complete Boolean algebra, and let X be a Banach space. Let $S, T : \mathscr{B} \longrightarrow X$ be σ-additive measures. If S is dividing and T is hereditarily dividing and has finite variation, then the pair S, T is uniformly dividing.*

It is an obvious observation that Theorem 6 yields Theorem 5. For the proof, we need several lemmas. Let $\mathscr{B}$ be a σ-complete Boolean algebra. A sequence $\tau = (a_n)_{n=1}^\infty$ in $\mathscr{B}$ is called a *tree* if $a_1 = \mathbf{1}$ and $a_n = a_{2n} \sqcup a_{2n+1}$ for all $n \in \mathbb{N}$. The minimal σ-complete subalgebra including a tree τ is called a *tree subalgebra* of $\mathscr{B}$ generated by τ.

Lemma 7. *Let $\tau = (a_n)_{n=1}^\infty$ be a tree in a σ-complete Boolean algebra $\mathscr{B}$ and let $\mathscr{U}_\tau$ be the tree subalgebra generated by τ. If $\mu : \mathscr{U}_\tau \longrightarrow [0, +\infty)$ is a σ-additive measure and $\mu(a_{2n+i}) \le (3/4)\mu(a_n)$ for all $n \in \mathbb{N}$ and $i \in \{0, 1\}$ then μ is atomless.*

Proof of Lemma 7. Fix any $a \in \mathscr{U}_\tau$ with $\mu(a) > 0$. By [14, Lemma 1.2.14], the smallest subalgebra $\mathscr{B}(\tau)$ of $\mathscr{B}_a$ including τ equals the set of all finite disjoint unions $p = \coprod_{i=1}^m a_{n_i}$ of elements of τ. By [15, 313F(c)], the σ-order closure of a subalgebra is a subalgebra. Hence, the σ-order closure of the subalgebra $\mathscr{B}(\tau)$ equals $\mathscr{U}$. Since μ is a σ-additive measure on $\mathscr{U}$, it is σ-continuous. Hence, we may and do choose a finite disjoint union $p = \coprod_{j \in J} a_j$ such that

$$\mu(a \triangle p) \le \frac{1}{6}\mu(a). \tag{1}$$

Set

$$p' = \coprod_{j \in J} a_{2j} \quad \text{and} \quad p'' = \coprod_{j \in J} a_{2j+1}. \tag{2}$$

Then

$$\mu(p') = \sum_{j \in J} \mu(a_{2j}) \le \frac{3}{4} \sum_{j \in J} \mu(a_j) = \frac{3}{4}\mu(p). \tag{3}$$

Similarly, $\mu(p'') \le (3/4)\mu(p)$. Hence, $\mu(p') = \mu(p) - \mu(p'') \ge (1/4)\mu(p)$. Thus,

$$\frac{1}{4}\mu(p) \le \mu(p') \le \frac{3}{4}\mu(p). \tag{4}$$

Observe that

$$\mu(a) \le \mu(p) + \mu(a \triangle p) \overset{\text{by (1)}}{\le} \mu(p) + \frac{1}{6}\mu(a). \tag{5}$$

Hence, $\mu(a) \le (6/5)\mu(p)$ and therefore, by (1), $\mu(a\triangle p) \le (1/5)\mu(p)$. Then we obtain

$$\begin{aligned}\mu(a \cap p') &= \mu(p') - \mu(p' - a) \ge \mu(p') - \mu(p \triangle a) \\ &\ge \frac{1}{4}\mu(p) - \frac{1}{5}\mu(p) = \frac{1}{20}\mu(p) \overset{\text{by (1)}}{>} 0.\end{aligned} \tag{6}$$

Similarly, $\mu(a \cap p'') > 0$. Hence

$$0 < \mu(a \cap p') < \mu(a \cap p') + \mu(a \cap p'') \le \mu(a), \tag{7}$$

which yields that a is not an atom for μ. □

Lemma 8. *Let $S : \mathscr{B} \longrightarrow X$ be a dividing σ-additive measure and let $\mu : \mathscr{B} \longrightarrow [0, +\infty)$ be a σ-additive atomless measure. Then there is a tree subalgebra $\mathscr{U}$ of $\mathscr{B}$ such that $S|_{\mathscr{U}}$ is a rank-one atomless measure and $\mu|_{\mathscr{U}}$ is an atomless measure.*

Proof of Lemma 8. First we prove the following *claim*: for every $b \in \mathscr{B}$ there exist $b', b'' \in \mathscr{B}$ such that $b = b' \sqcup b''$, $S(b') = S(b'') = (1/2)S(b)$, and $\mu(b'), \mu(b'') \le (3/4)\mu(b)$.

Using the atomlessness of μ, we choose a partition $b = b_1 \sqcup b_2$ with $\mu(b_1) = \mu(b_2)$ (formally we can apply [12, Theorem 2.11] to get this). Using the fact that S is dividing, we choose partitions $b_i = b_i' \sqcup b_i''$ so that $S(b_i') = S(b_i'') = (1/2)S(b)$ for $i = 1, 2$. With no loss of generality, we may and do assume that $\mu(b_i') \le \mu(b_i'')$ for $i = 1, 2$. Then $\mu(b_i') \le (1/2)\mu(b_i)$ for $i = 1, 2$. Set $b' = b_1' \sqcup b_2''$ and $b'' = b_1'' \sqcup b_2'$. Then

$$S(b') = S(b_1') + S(b_2'') = \frac{1}{2}(S(b_1) + S(b_2)) = \frac{1}{2}S(b) \tag{8}$$

and similarly $S(b'') = (1/2)S(b) = S(b')$. Moreover,

$$\begin{aligned}\mu(b') &= \mu(b_1') + \mu(b_2'') \le \frac{1}{2}\mu(b_1) + \mu(b_1) = \frac{3}{2}\mu(b_1) \\ &= \frac{3}{4}\mu(b).\end{aligned} \tag{9}$$

Similarly, $\mu(b'') \le (3/4)\mu(b)$. Hence

$$\mu(b') = \mu(b) - \mu(b'') \ge \mu(b) - \frac{3}{4}\mu(b) = \frac{1}{4}\mu(b), \tag{10}$$

which completes the proof of the claim.

To prove the lemma, we set $b_1 = \mathbf{1}$. Assume for a given $n \in \mathbb{N}$ that b_n has been already defined. Using the claim with $b = b_n$, we choose $b_{2n} = b'$ and $b_{2n+1} = b''$ such that $b_n = b_{2n} \sqcup b_{2n+1}$;

$$\begin{gathered} S(b_{2n}) = S(b_{2n+1}) = \frac{1}{2}S(b_n) \\ \text{and } \mu(b_{2n+i}) \le \frac{3}{4}\mu(b_n) \\ \text{for all } n \in \mathbb{N} \text{ and } i \in \{0, 1\}. \end{gathered} \tag{11}$$

Let $\mathscr{U}_\tau$ be the tree subalgebra generated by $\tau = (b_n)_{n=1}^{\infty}$. Observe that the image $S(\mathscr{B}(\tau))$ (the subalgebra $\mathscr{B}(\tau)$ was defined in the proof of Lemma 7) is the set of all vectors of the form

$$\begin{gathered}\left(\frac{\ell_1}{2^{k_1}} + \dots + \frac{\ell_j}{2^{k_j}}\right) S(b), \\ \text{where } j \in \mathbb{N}, \ \frac{\ell_1}{2^{k_1}} + \dots + \frac{\ell_j}{2^{k_j}} \in [0, 1],\end{gathered} \tag{12}$$

and so one has that $S(\mathscr{U}_\tau) = \{tS(t) : t \in [0, 1]\}$. Hence, there exists a scalar probability measure (i.e., σ-additive with nonnegative values and maximal value 1) $\mu_0 : \mathscr{U}_\tau \longrightarrow [0, 1]$ such that

$$\forall x \in \mathscr{U}_\tau, \quad S(x) = \mu_0(x) \cdot S(\mathbf{1}). \tag{13}$$

In particular, for every $n \in \mathbb{N}$, one has $\mu_0(b_n) = 2^{-k}$, where $n = 2^k + \ell$ with $\ell < 2^k$.

Finally, by Lemma 7, both scalar nonnegative measures $\mu|_{\mathscr{U}}$ and μ_0 are atomless, and hence $S|_{\mathscr{U}}$ is atomless. □

The following two lemmas seem to be well known.

Lemma 9. *Let $\mathscr{B}$ be a σ-complete Boolean algebra, and let X be a Banach space. Then every σ-additive finite rank measure $\nu : \mathscr{B} \longrightarrow X$ has finite variation $|\nu| : \mathscr{B} \longrightarrow [0, +\infty)$ which is σ-additive as well.*

To prove Lemma 9, one can use Hahn's decomposition theorem [15, 326 I] to every coordinate of an $\mathbb{R}^n$-valued measure and decompose unity of $\mathscr{B}$ into 2^n disjoint parts where every coordinate has a certain constant sign. Obviously, on every such a part ν has finite variation, which in their disjoint union gives $|\nu|$.

Lemma 10. *Let $\mathscr{B}$ be a σ-complete Boolean algebra, let X be a Banach space, and let $\nu : \mathscr{B} \longrightarrow X$ be a σ-additive measure having finite variation $|\nu| : \mathscr{B} \longrightarrow [0, +\infty)$. Then ν is atomless if and only if $|\nu|$ is.*

Proof of Lemma 10. Let $|\nu|$ be atomless. Assume, on the contrary, that there is an atom $a \in \mathscr{B}$ of ν. Then $\nu(a) \ne 0$ and $|\nu|(a) > 0$. Choose $b \le a$ so that $0 < |\nu|(b) < |\nu|(a)$. Then for every finite partition $a = \coprod_{k=1}^{m} a_k$ one has $\nu(a_{k_0}) = \nu(a)$ for some $k_0 \in \{1, \dots, m\}$ and $\nu(a_k) = 0$ for $k \ne k_0$ and hence $\sum_{k=1}^{m} \|\nu(a_k)\| = \|\nu(a)\|$. By the arbitrariness of the partition, $|\nu|(a) = \|\nu(a)\|$. Since $|\nu|(b) > 0$, there is $c \le b$ with $\nu(c) \ne 0$, and, hence, $\nu(c) = \nu(a)$ as a is an atom and $c \le a$. Thus, $\|\nu(a)\| \le |\nu|(c) \le |\nu|(b) < |\nu|(a) = \|\nu(a)\|$, a contradiction. □

Let ν be atomless. Let $a \in \mathscr{B}$ be such that $|\nu|(a) > 0$. Choose any $b \le a$ with $\nu(b) \ne 0$. Using the atomlessness of ν, we choose a partition $b = c \sqcup d$ so that $\nu(c) \ne 0 \ne \nu(d)$. Hence, $|\nu|(c) > 0$ and $|\nu|(d) > 0$, which implies that $0 < |\nu|(c) < |\nu|(a)$, and so a is not an atom for $|\nu|$. Thus, $|\nu|$ is atomless as well.

Proof of Theorem 6. Fix any $0 < b \in \mathscr{B}$. We let $\mu = |T|_{\mathscr{B}_b}|$. By Lemma 10, μ is atomless. Using Lemma 8 for $S|_{\mathscr{B}_b}$ (which is dividing and σ-additive as well) and μ, we choose a tree subalgebra $\mathscr{U}$ of $\mathscr{B}_b$ such that $S|_{\mathscr{U}}$ is a rank-one measure and $\mu|_{\mathscr{U}}$ is an atomless measure.

Show that the measures $T|_{\mathscr{U}}$ and $|S|_{\mathscr{U}}|$ are well defined and satisfy the assumptions of Lemma 8. Since $\mu|_{\mathscr{U}}$ is atomless, the measure $T|_{\mathscr{U}}$ is atomless as well by Lemma 10. And since T is hereditarily dividing, $T|_{\mathscr{U}}$ is dividing. By Lemma 8, the measure $S|_{\mathscr{U}}$ is atomless. By Lemma 9, the measure $|S|_{\mathscr{U}}|$ is well defined, and, by Lemma 10, $|S|_{\mathscr{U}}|$ is atomless.

Applying Lemma 8 to the measures $T|_{\mathscr{U}}$ and $|S|_{\mathscr{U}}|$, we choose a tree subalgebra $\mathscr{U}_1$ of $\mathscr{U}$ such that $T|_{\mathscr{U}_1}$ is a rank-one measure and $\mu|_{\mathscr{U}_1}$ is an atomless measure. Now consider the measure $V = (S|_{\mathscr{U}_1}, T|_{\mathscr{U}_1})$ which takes values in a 2-dimensional linear space $Y \times Z$, where Y and Z are the 1-dimensional subspaces of X in which the measures $S|_{\mathscr{U}_1}$ and $T|_{\mathscr{U}_1}$ take values, respectively. Since both coordinates are atomless measures, the measure V is atomless as well. Indeed, let $a \in \mathscr{U}_1$ be such that $V(a) \neq 0$, say, $S(a) \neq 0$. Then we choose $a' \leq a$ and $a' \in \mathscr{U}_1$ so that $0 \neq S(a') \neq S(a)$ and obtain that $0 \neq V(a') \neq V(a)$.

By Theorem 4, we can decompose $b = b' \sqcup b''$ so that $V(b') = V(b'')$; that is, $S(b') = S(b'')$ and $T(b') = T(b'')$. □

Using Theorem 4, Lemma 9 and Theorem 6, we obtain the following partial result.

Theorem 11. *Let $\mathscr{B}$ be a σ-complete Boolean algebra, and let X be a Banach space. Let $S, T : \mathscr{B} \longrightarrow X$ be σ-additive measures. If S is dividing and T is atomless and finite rank, then the pair S, T is uniformly dividing. In particular, $S + T$ is dividing.*

3. Implications to Orthogonally Additive Operators on Riesz Spaces

As mentioned in Introduction, there are many results on the narrowness of the sum of two narrow operators. Remark that all of them have common scheme of the proof: to prove that $S+T$ is narrow, it is sufficient to prove that every $e \in E$ admits a decomposition $e = e' \sqcup e''$ such that both vectors $S(e') - S(e'')$ and $T(e') - T(e'')$ are small in certain sense depending on the kind of narrowness.

Let E be an atomless Riesz space, and let X be a Banach space. We say that a pair of OAOs $S, T : E \longrightarrow X$ is *uniformly strictly narrow* if every $e \in E$ admits a decomposition $e = e' \sqcup e''$ such that $S(e') = S(e'')$ and $T(e') = T(e'')$. For the first time, the uniform narrowness of operators was considered in [16].

Recall that a net $(x_\alpha)_{\alpha \in \Lambda}$ in a Riesz space E *order converges* to an element $x \in E$ (notation $x_\alpha \xrightarrow{\mathrm{o}} x$) if there exists a net $(u_\alpha)_{\alpha \in \Lambda}$ in E such that $u_\alpha \downarrow 0$ and $|x_\beta - x| \leq u_\beta$ for all $\beta \in \Lambda$. A net (x_α) in E *laterally converges* to $x \in E$ if $x_\alpha \sqsubseteq x_\beta \sqsubseteq x$ for all indices $\alpha < \beta$ and $x_\alpha \xrightarrow{\mathrm{o}} x$. In this case we write $x_\alpha \xrightarrow{\ell} x$. For positive elements x_α, x the condition $x_\alpha \xrightarrow{\ell} x$ means that $x_\alpha \sqsubseteq x$ and $x_\alpha \uparrow x$.

Let E be a Riesz space and let X be a Banach space. An OAO $T : E \longrightarrow X$ is said to be *laterally-to-norm* continuous provided for every net (x_α) in E and every $x \in E$ the condition $x_\alpha \xrightarrow{\ell} x$ implies $\|T(x_\alpha) - T(x)\| \longrightarrow 0$. We say that an OAO $T : E \longrightarrow X$ has *finite variation* if for every $e \in E$

$$\sup \left\{ \sum_{k=1}^{m} \|T(e_k)\| : m \in \mathbb{N},\ e = \coprod_{k=1}^{m} e_k \right\} < \infty. \tag{14}$$

It is not hard to see that if E is Dedekind complete and $T : E \longrightarrow X$ is a laterally-to-norm continuous OAO then for every $e \in E$ the restriction $T|_{\mathfrak{F}_e}$ of T to the Boolean algebra $\mathfrak{F}_e$ of all fragments of e is a σ-additive measure. If, moreover, T has finite variation then the measure $T|_{\mathfrak{F}_e}$ is of finite variation as well for all $e \in E$.

For the proof of our main results, we need one more known lemma (see [12, Lemma 2.13]).

Lemma 12. *Let E be an atomless Dedekind complete Riesz space, let X be a Banach space, and let $T : E \longrightarrow X$ be a laterally-to-norm continuous OAO. Then for every $0 \neq e \in E$ the measure $T|_{\mathfrak{F}_e}$ is atomless.*

Remark that the original Lemma 2.13 from [12] is proven for positive elements $e > 0$ only. However, the general case then easily follows from the decomposition $e = e^+ - e^-$ and the observation that the same arguments work for negative elements.

Now an application of Theorem 11 gives the following result.

Theorem 13. *Let E be a Dedekind complete atomless Riesz space, let X be a Banach space, and let S, T be laterally continuous OAOs. If S is strictly narrow and T has finite rank, then $S+T$ is strictly narrow. Moreover, the pair S, T is uniformly strictly narrow.*

In order to apply Theorem 6, we first give a definition of a hereditarily strictly narrow operator. We say that an OAO $T : E \longrightarrow X$ is *hereditarily strictly narrow* if for any element $e \in E$ the restriction $T|_{\mathfrak{F}_e}$ is a hereditarily divisible measure.

As a consequence of Theorem 6 we obtain the following result.

Theorem 14. *Let E be a Dedekind complete atomless Riesz space, let X be a Banach space, and let S, T be laterally continuous OAOs. If S is strictly narrow and T is hereditarily strictly narrow and has finite variation, then $S + T$ is strictly narrow. Moreover, the pair S, T is uniformly strictly narrow.*

We conjecture that the assumption on T to have finite variation is superfluous. However, not is the sense that every hereditarily strictly narrow has finite variation (as it happened with finite rank operators), because this is not true as the following example shows.

Example 15. There exists a Dedekind complete atomless Riesz space E, a Banach space X, and a hereditarily strictly narrow linear bounded operator $T : E \longrightarrow X$ having infinite variation.

Construction. Let $1 < p < +\infty$. We set $E = X = L_p[0,1]$. Consider a disjoint sequence (A_n) of measurable subsets of $[0,1]$ with $[0,1] = \coprod_{n=1}^{\infty} A_n$ and $\mu(A_n) = n^{-p}(\sum_{j=1}^{\infty} j^{-p})^{-1}$ for all $n \in \mathbb{N}$. Then the conditional expectation operator with respect to the σ-algebra generated by A_ns

$$Tx = \sum_{n=1}^{\infty} \frac{1}{\mu(A_n)} \left(\int_{A_n} x \, d\mu \right) \mathbf{1}_{A_n}, \tag{15}$$

where $\mathbf{1}_A$ is the characteristic function of A, possesses the desired properties.

Remark that we are still far from a solution of Problem 2. Another related problem is in [16] and is still unsolved.

Problem 16. Let E be a Riesz space and let X be a Banach space (or, more generally, F-space). Are the following assertions equivalent for every pair of narrow linear operators (or, more generally, OAOs) $S, T : E \longrightarrow X$?

(i) $S + T$ is narrow;

(ii) S, T are uniformly narrow.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors are grateful to V. Kadets for valuable remarks. This research was supported by the University of Silesia Mathematics Department (Iterative Functional Equations and Real Analysis program).

References

[1] A. M. Plichko and M. M. Popov, "Symmetric function spaces on atomless probability spaces," *Dissertationes Mathematicae*, vol. 306, pp. 1–85, 1990.

[2] M. Popov and B. Randrianantoanina, *Narrow operators on function spaces and vector lattices*, vol. 45 of *De Gruyter Studies in Mathematics*, De Gruyter, 2013.

[3] V. V. Mykhaylyuk and M. M. Popov, "On sums of narrow operators on Köthe function spaces," *Journal of Mathematical Analysis and Applications*, vol. 404, no. 2, pp. 554–561, 2013.

[4] I. V. Krasikova, "A note on narrow operators in L∞," *Matematychni Studii*, vol. 31, no. 1, pp. 102–106, 2009.

[5] V. Mykhaylyuk, "On the sum of a narrow and a compact operators," *Journal of Functional Analysis*, vol. 266, no. 9, pp. 5912–5920, 2014.

[6] M. Pliev and M. Popov, "Narrow orthogonally additive operators," *Positivity. An International Mathematics Journal Devoted to Theory and Applications of Positivity*, vol. 18, no. 4, pp. 641–667, 2014.

[7] J. M. Mazón and S. Segura de León, "Order bounded ortogonally additive operators," *Revue Roumaine de Mathématique Pures et Appliquées*, vol. 35, no. 4, pp. 329–353, 1990.

[8] J. M. Mazón and S. Segura de León, "Uryson operators," *Revue Roumaine des Mathematiques Pures et Appliquees*, vol. 35, no. 5, pp. 431–449, 1990.

[9] O. V. Maslyuchenko, V. V. Mykhaylyuk, and M. M. Popov, "A lattice approach to narrow operators," *Positivity. An International Mathematics Journal Devoted to Theory and Applications of Positivity*, vol. 13, no. 3, pp. 459–495, 2009.

[10] O. Fotiy, A. Gumenchuk, I. Krasikova, and M. Popov, "On sums of narrow and compact operators," *Positivity*, pp. 1–12, 2019.

[11] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Springer, Dordrecht, The Netherlands, 2006.

[12] V. Mykhaylyuk, M. Pliev, M. Popov, and O. Sobchuk, "Dividing measures and narrow operators," *Studia Mathematica*, vol. 231, no. 2, pp. 97–116, 2015.

[13] V. Kadets, N. Kalton, and D. Werner, "Unconditionally convergent series of operators and narrow operators on L_1," *Bulletin of the London Mathematical Society*, vol. 37, no. 2, pp. 265–274, 2005.

[14] V. I. Bogachev, *Measure Theory*, vol. 1, Springer, Berlin, Germany, 2007.

[15] D. H. Fremlin, *Measure Theory*, vol. 3 of *Measure algebras*, Torres Fremlin, Colchester, UK, 2004.

[16] A. I. Gumenchuk, I. V. Krasikova, and M. M. Popov, "Points of narrowness and uniformly narrow operators," *Carpathian Mathematical Publications*, vol. 9, no. 1, pp. 37–47, 2017.